PROBLEM SOLUTIONS, EDINBURGH BEng/MEng/MSc COURSE - 2007

DIGITAL COMMUNICATIONS Second Edition

I A Glover and P M Grant

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SELECTED PROBLEM SOLUTIONS CHAPTER 3

3.3(a)

As there are no discontinuities in the cdf

$$a(1 + \sin(-2b)) = 0 \quad (\textit{from } x = -2)$$

$$\Rightarrow \; b = \frac{\pi}{4}$$

$$F_x(\infty) = 1 \Rightarrow c = 1$$

and finally,

$$a\left(1 + \sin\left(\frac{\pi}{2}\right)\right) = 1 \Rightarrow a = ½$$

3.3(b)

$$P(X < 0) = P_X(0) = ½$$

3.3(c)

$$p_X(x) = \frac{d}{dx} P_X(x)$$

$$\frac{d}{dx}\left\{\frac{1}{2}\left(1 + \sin\left(\frac{\pi x}{4}\right)\right)\right\} = \frac{\pi}{8}\cos\left(\frac{\pi x}{4}\right)$$

Thus

$$p_X(x) = \begin{cases} \frac{\pi}{8}\cos\left(\frac{\pi x}{4}\right), & |x| \le 2 \\ 0, & |x| > 2 \end{cases}$$

3.4(a)

$P(X > 0.5) = 1 - P(X \le 0.5) = 1 - P_X(0.5) = 0.6065$

3.4(b)

$P(X \le 0.25) = P_X(0.25) = 0.2212$

3.4(c)

$P(0.3 < X \le 0.7) = P(X \le 0.7) - P(X \le 0.3)$

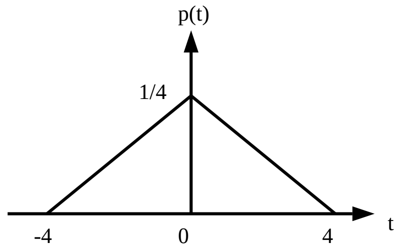
$$= P_X(0.7) - P_X(0.3) = 0.2442$$

3.5

$$P(w > w_0) = \frac{1}{w_0} \int_{w_0}^{\infty} e^{-\frac{w}{w_0}} dw = \left[w_0 \frac{1}{w_0} e^{-\frac{w}{w_0}} \right]_{w_0}^{\infty} = e^{-1} = 0.3679$$

3.6(a)

Distribution with the mean ($\bar{X}$) removed:



$$\sigma^2 = \int_{-\infty}^{\infty} t^2 p(t)\, dt = \int_{-4}^{0} t^2\left(\frac{1}{16}t + \frac{1}{4}\right) dt + \int_{0}^{4} t^2\left(-\frac{1}{16}t + \frac{1}{4}\right) dt = 2.66$$

3.6(b)

Standard deviation, $\sigma = 1.63$

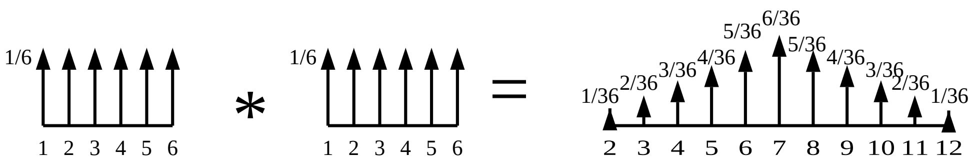
3.6(c)

Reject all those greater than 10 or, removing the mean, all those greater than 2

$$= \int_{2}^{4} \left(-\frac{1}{16}t + \frac{1}{4}\right) dt = 12.5\%$$

3.11

(a)



(b) Answer is obtained by convolving the answer to part a) with itself with a shift of 24, i.e.:

$$\left(\frac{1}{36}\right)^2 = 7.7\times10^{-4}$$

(c) The most probable sum for four die is when the answer to part a) is convolved with itself such that the two distributions are lined up. This occurs with a shift of 14, so 14 is the most probable sum and the probability of 14 is given by

$$2\left(\frac{1}{36}\right)^2 + 2\left(\frac{2}{36}\right)^2 + 2\left(\frac{3}{36}\right)^2 + 2\left(\frac{4}{36}\right)^2 + 2\left(\frac{5}{36}\right)^2 + \left(\frac{6}{36}\right)^2 = 0.113$$

(d)

number of die (n)	2	4	100
minimum total = n	2	4	100
maximum total = $6n$	12	24	600
average = min+(min+max)/2	7	14	350
number of possible sums = (max-min)+1	11	21	501

The central limit theorem says that the distribution becomes a better and better approximation to Gaussian as more and more die are added.

SELECTED PROBLEM SOLUTIONS CHAPTER 7

7.2

This occurs when the losses and a priori probabilities are equal.

7.3

As the losses and a priori probabilities are equal, we should choose '1' if $P(v|1) > P(v|0)$.
Threshold will be at 0 V.
$P_e(total) = P(1)P(v < 0|1) + P(0)P(v > 0|0)$
For an erf lookup where $m = \bar{X}$:

$$z = \frac{x-m}{\sqrt{2}\sigma} = \frac{0-5}{\sqrt{2}\times 2} = -1.767767 \quad \text{for data bit 1}$$

$$= +1.767767 \quad \text{for data bit 0}$$

Now for only 1/0 transmissions as in Question 6.10:

$$P_e(1) = (½ - ½\text{erf}(1.767767)) = (½ - ½\times 0.98758067)$$

$$= 0.0062097$$

$$P_e(0) = P_e(1)$$

Now total $P_e = ½P_e(1) + ½P_e(0)$. Thus:

$$P_e(total) = 0.0062097$$

7.4

This is an example of a maximum likelihood receiver: $P_e(total) = P(1)P(v < 0|1) + P(0)P(v > 0|0)$
For the normalised Gaussian, x is changed from 0 to 0.5 with respect to Problem 7.3:

$$z = \frac{x-m}{\sqrt{2}\sigma} = \frac{0.5-5}{\sqrt{2}\times 2} = -1.5909903 \quad \text{for data bit 1}$$

$$z = \frac{x-m}{\sqrt{2}\sigma} = \frac{0.5+5}{\sqrt{2}\times 2} = 1.9445436 \quad \text{for data bit 0}$$

$$P_e(1) = (½ - ½\text{erf}(1.5909903)) = (½ - ½\times 0.97555106) = 0.0122245$$

$$P_e(0) = ½ - ½\text{erf}(1.9445436) = 1 - (½ + ½\times 0.99404047) = 0.0029798$$

$$P_e(total) = 0.0076022$$

Note increase over Problem 7.3!

7.5.

Using Bayes's decision rule:

$$\frac{p(v_{th}|1_{TX})P(1_{TX})}{p(v_{th}|0_{TX})P(0_{TX})} > \frac{L_1}{L_0}$$

$$L_0 = 1,\ L_1 = 3,\ P(1) = 2/3,\ P(0) = 1/3,\ p(v_{th}|1) = \frac{1}{\sqrt{2\pi}}\,e^{-\frac{(v_{th}-1)^2}{2}},\ p(v_{th}|0) = \frac{1}{\sqrt{2\pi}}\,e^{\frac{-v_{th}^2}{2}}$$

$$\Rightarrow \frac{e^{-\frac{(v_{th}-1)^2}{2}}}{e^{\frac{-v_{th}^2}{2}}} = 1.5$$

$$e^{\left(-\frac{v_{th}^2}{2}+v_{th}-1/2+\frac{v_{th}^2}{2}\right)} = 3/2$$

$$\log_e(3/2) = v_{th} - 1/2 \Rightarrow v_{th} = 0.905$$

7.6

For a probability of false alarm of 0.01

$$1 - (½ + ½\mathrm{erf}(z)) = 10^{-2}$$

$$\mathrm{erf}(z) = 0.9800$$

Thus $z = 1.645 = x/(\sqrt{2}\sigma)$. $\sigma = \sqrt{5} = 2.236$, therefore x (the threshold) = 5.202.
The probability of a target of 4 being detected is calculated via:

$$z = \frac{x - m}{\sqrt{2}\sigma} = \frac{5.21 - 4}{\sqrt{2}\times 2.24} = 0.3801$$

$$P_e(1) = 1 - (½ + ½\mathrm{erf}(0.3801))$$

$$= 0.2954$$

SELECTED PROBLEM SOLUTIONS CHAPTER 5

5.0

3.4 kHz signal has a minimum practical sample rate of 2.2×3.4 kHz i.e. 7.48 ksample/s. We normally round up to an 8 kHz sample rate to give 8 kHz × 8 bit words = 64 kbit/s overall rate in telecomms systems.

5.00

Nonlinear companding does not alter the sample rate! For an 8 kHz sample rate 8 kHz × 10 bit words = 80 kbit/s overall bit rate.

5.1(a)

FDM spectrum has signals occupying different frequency slots while in a TDM time domain signal the samples occur in sequence. The FDM thus consists of a parallel bank of modulators whose outputs are summed and the TDM system comprises parallel bank of samplers followed by a commutating switch.

5.1(b)

In PAM input sample amplitude maps to transmitted pulse height. In PPM the amplitude maps to precise pulse position and in PCM we apply an ADC to code the sample amplitudes as short sequences of binary digit words.

5.1(c)

FDM bandwidth with SSB modulators = 4×12 = 48 kHz. TDM bandwidth is related to sample rate which is 12×8 kHz = 96 ks/s. We say TDM-PAM thus occupies 48 kHz bandwidth. TDM-PCM with 8-bit encoder occupies 48×8 = 384 kHz. TDM-PPM with 2% resolution requires the sample time of $1/(96 \times 10^3)$ = 10.5 μs to be split into 50 timeslots. Thus there is 1/5 μs per slot and the -3 dB bandwidth of the pulse will be 2.5 MHz.

5.2

Sample rate per channel = 10 kHz. Multiplexed sample rate = 20 kHz, thus clock rate is 20 kHz. Filter bandwidth is ½ clock rate, i.e. 10 kHz.

5.7

(a) TDM pulse rate $R_b = 25 \times 8$ kHz = 200 kHz

$$\text{Bandwidth} = \frac{R_b}{2} = \frac{200}{2} = 100 \text{ kHz}$$

(b) For resolution of 5% we need 20 distinguishable positions for each pulse. Therefore each pulse must be $\frac{1}{20}$ the width of the pulses in part(a) and the bandwidth is therefore 20 times the bandwidth in part(a)

i.e. Bandwidth = 20×100 kHz = 2.0 MHz

(c) For a resolution of 0.5% we need $M > 200$ distinguishable levels. Number of bits per PCM codeword is therefore:

$$n > \log_2 200 = 7.6$$

i.e. $n = 8$ bits/code word

PCM bit rate, $R_b = n \times 8\text{kHz} \times 25 = 1600$ kHz

$$\text{Bandwidth} = \frac{R_b}{2} = \frac{1600}{2} \text{ kHz} = 800 \text{ kHz}$$

5.8

$$f_s = 2\ f_H = 2 \times 20 \text{ kHz} = 40 \text{ kHz}$$

From equation (5.23):

$$\text{SN}_q\text{R} = 4.8 + 6n - \alpha_{dB}$$

$$\therefore n = \frac{\text{SN}_q\text{R} - 4.8 + \alpha_{dB}}{6}$$

$$= \frac{55.0 - 4.8 + 20.0}{6}$$

$$= 12 \quad \text{(since } n \text{ must be an integer)}$$

$$R_b = f_s\, n$$

$$= 40 \times 10^3 \times 12$$

$$= 480 \times 10^3 \text{ bit/s}$$

$$= 480 \text{ kbit/s}$$

Minimum baseband bandwidth required for (ISI free) transmission of this bit rate is given by:

$$B = \frac{R_b}{2} = \frac{480}{2} \text{ kbit/s} = 240 \text{ kHz}$$

5.10

For this problem you can just assume the formula (the full solution follows later) and:
For U.S.A. $\mu = 255$ and $n = 8$

$$\frac{S}{N} = \frac{3 \times 2^{2n}}{[\ln(1+\mu)]^2} = \frac{3 \times 2^{16}}{(\ln 256)^2}$$

$$= \frac{3 \times 2^{16}}{(\log_e 10 \times \log_{10} 256)^2}$$

$$= \frac{3 \times 2^{16}}{(2.3 + 2.4)^2}$$

$$\therefore \frac{S}{N} = 6400$$

$$\equiv 38 \text{ dB}$$

For linear PCM $n = 8$ and $\frac{S}{N} = 3 \times 2^{2n}$. Thus in dB $\frac{S}{N} = 53$ dB

$\therefore$ Degradation = 53 − 38 = 15 dB

or more accurate solution is:

For μ-law companding:

$$F(x) = \text{sgn}(x) \frac{\ln(1 + \mu|x|)}{\ln(1+\mu)}$$

Restricting attention to $x > 0$ only:

$$\frac{dF(x)}{dx} = \frac{1}{\ln(1+\mu)} \frac{d}{dx}\left[\ln(1+\mu x)\right]$$

$$= \frac{1}{\ln(1+\mu)}\left[\frac{1}{1+\mu x}\right]\mu$$

$$= \frac{\mu}{1+\mu x}\ \frac{1}{\ln(1+\mu)}$$

For largest signals (i.e. $x = 1$)

$$\frac{dF(x)}{dx} = \frac{\mu}{1+\mu}\ \frac{1}{\ln(1+\mu)}$$

and for typical value of μ (e.g. $\mu = 255$):

$$\frac{dF(x)}{dx} = \frac{1}{\ln(1+\mu)}$$

This large signal gradient of μ-law is $\frac{1}{\ln(1+\mu)}$ times that of linear law. Peak signal to quantisation noise ratio for linear quantisation is given by equation (5.18), i.e.:

$$\begin{aligned} \mathrm{SN_qR}|_{linear} &= 3\,M^2 \\ &= 3\,(2^n)^2 \\ &= 3 \times 2^{2n} \end{aligned}$$

The large signal μ-law $\mathrm{SN_qR}$ is decreased by the factor $\left[\frac{dF(x)}{dx}\right]^2$ with respect to linear quantisation. Therefore:

$$\mathrm{SN_qR}|_{\mu-law} = \frac{3 \times 2^{2n}}{[\ln(1+\mu)]^2}$$

when $n = 8$

$$\begin{aligned} \text{Companded SNR} &= \frac{3 \times 2^{2\times 8}}{[\ln(1+255)]^2} \\ &= 6394 \\ &= 38.06\ \text{dB} \end{aligned}$$

$$\begin{aligned} \text{Uncompanded SNR} &= 3 \times M^2 - 1 \\ &= 3 \times (2^n)^2 - 1 \\ &= 3 \times 2^{8\times 2} - 1 \\ &= 196607 \\ &= 52.94\ \text{dB} \end{aligned}$$

Degradation = 52.94 – 38.06 = 14.88 dB

Note that this problem involves a PEAK SNR calculation & not the normal mean SNR one!

5.11

Calculate intersection points for the piecewise linear approximation, i.e. for $F(x)$ values for $|x| = \frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$ etc.:

$$F(x) = \frac{1 + \ln(87.6x)}{1 + \ln 87.6} = \frac{1 + \ln(87.6x)}{1 + (2.3 + 2.17)}$$

$$= \frac{1 + \ln(87.6x)}{5.47}$$

to obtain table of intersection points between segments in Figure 5.26 shown by the bold dots:

Segment no.		$F(x)$	$\lvert x\rvert$
		1.0	1.0
1	<		
		0.87	$\frac{1}{2}$
2	<		
		0.75	$\frac{1}{4}$
3	<		
		0.62	$\frac{1}{8}$
4	<		
		0.49	$\frac{1}{16}$
5	<		
		0.37	$\frac{1}{32}$
6	<		
		0.24	$\frac{1}{64}$
7 & 8	<		
		0.0	

There are another two steps from 0 to -0.24 then a further 6 steps from -0.24 to -1.0 to give 16 overall over the ±1V range but there are only 13 distinct slopes!

We allocate the 8 bit code as follows:
3 bits of code define segment location.
1 bit of code for sign (+/-).
4 bits for location on segment as 4 bits imply $2^4 = 16$ steps/segment.

For close to full scale signals we must calculate stepsize on top segment.

On top segment $[F(x)]$ lies between 0.87 and 1 (or 0.5 to 1 on the $|x|$ input). Thus input stepsize = $(1 - 0.5)/16 = 0.031$. This implies a total number of **equivalent** linear converter steps of 32 over x = 0 to 1 and 64 steps over the full ± 1 V range of input signal values. Thus a 64 step linear converter implies 6 bits and, at 6 dB/bit, a 36 dB SNR.

Now examine stepsize close to the origin.

On the 4 co-linear segments about the origin there are $2 \times 16 = 32$ steps to the origin giving an input voltage stepsize of $(0.0156 - 0)/32 = 0.000488$. (0.0156 = 1/64) This implies a total of 2049 steps if this stepsize were repeated over the 0 to 1V range.

Equivalent number of linear steps over the entire range (± 1 V) implies 12 bit linear converter with $12 \times 6 = 72$ dB SNR. (This stepsize implies 4098 steps over the ±1 V input voltage range and 2^{12} = 4096 implying an equivalent 12 bit linear converter for the same SNR performance!) Thus with *A*-law we degrade the SNR at large signal voltages but significantly improve SNR for small signals.

A more accurate analysis is:

For linear quantisation $\mathrm{SN}_q\mathrm{R}$ is given by:

$$\begin{aligned}\mathrm{SN}_q\mathrm{R} &= M^2 - 1 \\ &= (2^n)^2 - 1 \\ &= (2^8)^2 - 1 \\ &= 65535 \\ &= 48.2 \text{ dB}\end{aligned}$$

For small signals (in segments 7 and 8) the average gradient, $\dfrac{dF(x)}{dx}$, is given by:

$$\begin{aligned}\frac{dF(x)}{dx}\Big|_{small\ signals} = \frac{F\left(\frac{1}{64}\right)}{\left(\frac{1}{64}\right)} &= \left(\frac{1 + \ln\left(87.6 \times \frac{1}{64}\right)}{1 + \ln 87.6}\right) / \left(\frac{1}{64}\right) \\ &= \frac{0.24}{1/64} \\ &= 15.36\end{aligned}$$

Thus there will be 15.36 times more levels in segments 7 and 8 than there would be for linear quantisation. This represents an improvement in $\mathrm{SN}_q\mathrm{R}$ of $20 \log_{10}(15.26) = 23.7$ dB

Thus for small signals (segments 7 and 8):

$$\mathrm{SN}_q\mathrm{R} = 48.2 + 23.7 = 71.9 \text{ dB}$$

For large signals (segment 1) gradient is given by:

$$\begin{aligned}\frac{dF(x)}{dx}\Big|_{large\ signals} &= \frac{F(1) - F(½)}{½} \\ &= \frac{1 - \left(\frac{1 + \ln(87.6 \times ½)}{1 + \ln 87.6}\right)}{½} \\ &= \frac{1 - 0.873}{½} \\ &= 0.253\end{aligned}$$

Thus there will be 0.253 times less levels in segment 1 than there would be for linear quantisation representing a SN_qR degradation of $20\log_{10}(0.253) = -11.9$ dB, i.e. for large signals (segment 1):

$$\text{SN}_q\text{R} = 48.2 - 11.9 = 36.3 \text{ dB}$$

Thus the errors in the approximate analysis are only tenths of a dB!

SELECTED PROBLEM SOLUTIONS CHAPTER 6

6.10(a)

First we find z where $m = \bar{X}$:

$$z = \frac{x - m}{\sqrt{2}\sigma} = \frac{5.5 - 3}{\sqrt{2}\times 2} = 0.8839$$

Therefore probability that $x \le 5.5$ is:

$$½ + ½\text{erf}(0.8839) = ½ + ½\times 0.7887 = 0.89$$

6.10(b)

P_e is now the above value subtracted from 1, i.e. 0.11

6.10(c)

$$z = \frac{x - m}{\sqrt{2}\sigma} = \frac{2750 - 1830}{\sqrt{2}\times 460} = 1.41$$

Probability of the clouds being higher than 2750 metres is therefore:-

$$1 - (½ + ½\text{erf}(1.41)) = 1 - ½ - ½\times 0.954 = 0.023$$

6.10(d) (i)

This question assumes zero mean, therefore for the three cases:-

$$i)\ \ z = \frac{\sigma}{\sqrt{2}\sigma} = \frac{1}{\sqrt{2}}\,, \quad ii)\ \ z = \frac{2}{\sqrt{2}}\,, \quad iii)\ \ z = \frac{3}{\sqrt{2}}$$

This gives erf z as i) erf(0.71) = 0.6827, ii) erf (1.41) = 0.9544, iii) erf(2.12) = 0.9973. Therefore probability that the Gaussian random variable will exceed these values is given by:-

$$\text{i)}\ \ 1 - (½ + ½\times 0.6827) = 0.16$$

$$\text{ii)}\ \ 1 - (½ + ½\times 0.9544) = 0.023$$

$$\text{iii)}\ \ 1 - (½ + ½\times 0.9973) = 0.0013$$

SELECTED PROBLEM SOLUTIONS CHAPTER 9

9.1(a)

$I(x_1) = -\log_2\left(\frac{1}{2}\right) = 1$ bit.

$I(x_2) = I(x_3) = -\log_2\left(\frac{1}{4}\right) = 2$ bits.

9.1(b)

$I(x_i) = -\log_2\left(\frac{1}{2^K}\right) = K$ bits as there are 2^K possible sequences of length K.

9.1(c)

(i) $I(x_1 x_2 x_3 x_4) = -\log_2\left(\frac{1}{2}\,\frac{1}{8}\,\frac{1}{4}\,\frac{1}{2}\right) = 7$ bits.

(ii) $I(x_1 x_2 x_3 x_4) = -\log_2\left(\frac{1}{4}\,\frac{1}{4}\,\frac{1}{4}\,\frac{1}{4}\right) = 8$ bits.

9.2

(a) $H = \frac{1}{2}\,1 + \frac{1}{4}\,2 + \frac{1}{4}\,2 = 1.5$ bits/symbol.

(b) (i) $H = \frac{1}{2}\,1 + \frac{1}{4}\,2 + \frac{1}{8}\,3 + \frac{1}{8}\,3 = 1.75$ bits/symbol.

(ii) $H = \frac{1}{4}\,2 + \frac{1}{4}\,2 + \frac{1}{4}\,2 + \frac{1}{4}\,2 = 2$ bits/symbol.

This source has the maximum entropy as the symbols are all equiprobable.

(c) When all symbols are equiprobable $H_{\max} = \left(\frac{1}{8}\,3\right)8 = 3$ bits/symbol.

$H = \frac{1}{2}\,1 + \left(\frac{1}{8}\,3\right)3 + \left(\frac{1}{32}\,5\right)4 = 2.25$ bits/symbol and the redundancy is 0.75 bit/symbol.

9.6

(i) Signal to noise ratio = 0 dB $\Rightarrow x/\sigma = 1$, $z = x/\sqrt{2}\sigma = 0.7071$. As in Problem 6.10:

$P_e \;=\; 1 - (½ + ½\ \mathrm{erf}(0.7071)) = 1 - ½ - ½ \times 0.6829 = 0.1587$

Thus the probability of correct reception = 0.8413.
Loss in information per binary digit is thus obtained by calculating the transmitted information, equation (9.10), and subtracting the received information, equation (9.12), for *both* the 1 and 0 symbols:

$$I_{RX}\ (j_{RX}) \;=\; \log_2\ \frac{P\ (i_{TX}\mid j_{RX})}{P\ (i_{TX})} \qquad (9.12)$$

$$\bar{I}_{RX}\ (j_{RX}) \;=\; \sum_i\ P\ (i_{TX}\mid j_{RX}) \log_2\ \frac{P\ (i_{TX}\mid j_{RX})}{P\ (i_{TX})}$$

$$=\; 0.8413 \log_2\left(\frac{0.8413}{0.5}\right) \;+\; 0.1587 \log_2\left(\frac{0.1587}{0.5}\right)$$

$$=\; 0.3689 \ \text{bit/symbol}$$

$$H_{eff} \;=\; \sum_i P(i_{RX})\ \bar{I}_{RX}\ (i_{RX})$$

$$= (0.5 \times 0.3689) + (0.5 \times 0.3689)$$
$$= 0.3689 \text{ (bit/symbol)}$$

Information lost, $E = H - H_{eff}$

$$= 1 - 0.3689$$
$$= 0.6311 \text{ bit/symbol}$$

Alternatively, using equation (9.16):

$$E = \sum_j P(j_{RX}) \sum_i P(i_{TX} \mid j_{RX}) \log_2 \frac{1}{P(i_{TX} \mid j_{RX})} \qquad (9.16)$$

$$= P(0_{RX}) \left[P(0_{TX} \mid 0_{RX}) \log_2 \frac{1}{P(0_{TX} \mid 0_{RX})} + P(1_{TX} \mid 0_{RX}) \log_2 \frac{1}{P(1_{TX} \mid 0_{RX})} \right]$$

$$+ P(1_{RX}) \left[P(0_{TX} \mid 1_{RX}) \log_2 \frac{1}{P(0_{TX} \mid 1_{RX})} + P(1_{TX} \mid 1_{RX}) \log_2 \frac{1}{P(1_{TX} \mid 1_{X})} \right]$$

$$= 0.5 \left[0.8413 \log_2 \left(\frac{1}{0.8413} \right) + 0.1587 \log_2 \left(\frac{1}{0.1587} \right) \right]$$

$$+ 0.5 \left[0.1587 \log_2 \left(\frac{1}{0.1587} \right) + 0.8413 \log_2 \left(\frac{1}{0.8413} \right) \right]$$

$$= 0.5 [0.2097 + 0.4214] + 0.5 [0.4214 + 0.2097]$$

$$= 0.6311 \text{ bit/symbol}$$

(ii) 5 dB $\Rightarrow x/\sigma = 1.78$, $z = 1.2574$. Probability of correct reception = 0.9623. Loss in information = 1 – 0.7693 = 0.2307 bit/binary digit.

(iii) 10 dB $\Rightarrow x/\sigma = 3.16$, $z = 2.2345$. Probability of correct reception = 0.9992173. Loss in information = 1 – 0.9906 = 0.0094 bit/binary digit.

S.1 Information per pixel $=\log_2(64) = 6$. Information per frame = 6×625×550 = 2.0625 Mbit. Information rate = 2062500×25=51.5625 Mbit/s = R_{max}.

$$R_{max} = B \log_2 \left(1 + \frac{S}{N} \right)$$

$$B = \frac{R_{max}}{\log_2 \left(1 + \frac{S}{N} \right)} = \frac{51562500}{\log_2 (1 + 3162)} = 4.43 \text{ MHz}$$

9.5

Effective number of pixels per picture, N_{pix}, is given by:

$$N_{pix} = 625 \times \left(625 \times \frac{4}{3} \right) = 520800$$

For $M = 10$:

$$I_{pixel} = \log_2 10 = \frac{\log_{10} 10}{\log_{10} 2} = 3.322 \text{ bit/symbol}$$

This has to be rounded up to a whole number of bits (4) and then max information content of picture is therefore:

$$\begin{aligned} I_{picture} &= N_{pix} \times I_{pixel} \\ &= 520800 \times 4 \\ &= 2.0832 \text{ Mbit} \end{aligned}$$

For a picture rate of 25 Hz this data rate is:

$$\begin{aligned} R_b &= I_{picture} \times \text{picture rate} \\ &= 2.0832 \text{ Mbit} \times 25 \text{ Hz} \\ &= 52.08 \text{ Mbit/s} \end{aligned}$$

For binary baseband of transmission minimum ISI free bandwidth is given by:

$$B = \frac{1}{2T_o} = \frac{R_s}{2} = \frac{52.08 \times 10^6}{2} = 26.04 \text{ MHz}$$

9.7

For 100 different symbols we need binary code words of length L where:

$$\begin{aligned} L &\geq \log_2 100 \\ &= 6.644 \end{aligned}$$

i.e.:

$$L = 7 \text{ bits}$$

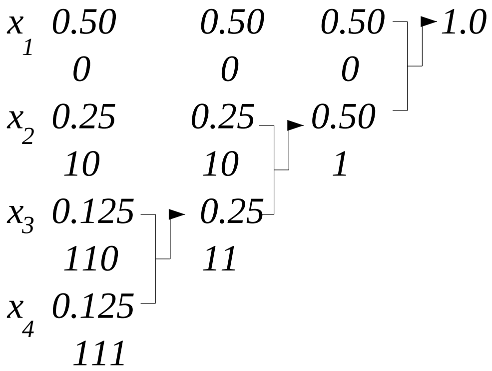
$$H = \log_2 100$$

From equation (9.21):

$$\begin{aligned} \eta_{code} &= \frac{H}{L} \times 100\% \\ &= \frac{6.644}{7} \times 100\% \\ &= 94.9\% \end{aligned}$$

9.8

(a)



$x_1 = 0$, $x_2 = 10$, $x_3 = 110$ and $x_4 = 111$. Code is uniquely decodable.

(b) Efficiency = $\frac{\text{entropy}}{\text{average codeword length}} = \frac{H}{L_{av}} \times 100\%$.

Average length $L_{av} = \frac{1}{2}1 + \frac{1}{4}2 + \left(\frac{1}{8}3\right)2 = 1.75$

$H = \frac{1}{2}1 + \frac{1}{4}2 + \left(\frac{1}{8}3\right)2 = 1.75$

Efficiency is 100% as probabilities are $\frac{1}{2^m}$.

(c) $x_1 = 00$, $x_2 = 01$, $x_3 = 10$ and $x_4 = 11$ for example.

efficiency$=\frac{1.75}{2} = 87.5\%$.

9.9

Coding scheme consists of three fields for a match:
1 xxxx xx
First field is a match flag, 1 for a match. Second field is the position of the start of the match, 0000 for the last character transmitted and 1111 for 16 characters ago. The third field is the length of a match, 00 for 1 character (a match length of 0 makes no sense) and 11 for 4 characters.

If there is no match we have two fields:
0 xxxxxxxx
the first field is the match flag, 0 for no match. The second field is the eight bit ASCII code for the character.

To be transmitted	History
1415927	1414213617920408

The 7 encodes as 1 1001 00 (7 bits). We now have:

To be transmitted	History
141592	7141421361792040

The 92 encodes as 1 1011 01 (7 bits). We now have:-

To be transmitted	History
1415	9271414213617920

The 5 encodes as 0 00110101 where 00110101 is the ASCII code for 5 (9 bits). We now have:-

To be transmitted	History
141	5927141421361792

141 encodes as 1 0100 10 (7 bits).

We have encoded 7 eight bit characters (56 bits) into 30 bits, a compression factor of 30/56 = 0.536.

SELECTED PROBLEM SOLUTIONS CHAPTER 10

10.1

(a) The average number of errors per block = $20 \times 0.05 = 1$.

(b) Probability of a symbol error is the probability that more than 3 errors occur in a block of 20 bits.

$$= 1 - P(0\ errors) - P(1\ error) - P(2\ errors) - P(3 errors)$$

$$= 1 - (1 - P_e)^{20} - P_e^1(1 - P_e)^{19}\ {}^{20}C_1 - P_e^2(1 - P_e)^{18}\ {}^{20}C_2 - P_e^3(1 - P_e)^{17}\ {}^{20}C_3$$

$$= 1 - (0.95)^{20} - 0.05(0.95)^{19}20 - (0.05)^2(0.95)^{18}\frac{20!}{2!18!} - (0.05)^3(0.95)^{17}\frac{20!}{3!17!}$$

$$= 1 - 0.3585 - 0.3774 - 0.1887 - 0.0596 = 0.0158$$

At 20,000 binary digits per second we have 1000 symbols per second. Symbol error rate =1000 × 0.0158=15.8 errors/s.

10.2

For the uncoded case, probability of error in a block of 4 data bits as in the example is

$$1.0 - (1.0 - P_e)^4 = 0.004$$

From erf tables, the normalised value of z which gives 0.9990 is again given by:

$(½ + ½\ \mathrm{erf}\ (z)) = 0.9990$

$\mathrm{erf}(z) = 0.9980$

$z = 2.186$

Therefore as in Problems 7.4 and 7.6

$$z = \frac{x - m}{\sqrt{2}\sigma}$$

$x/\sigma = 3.09$ is the voltage signal to noise ratio and $(x/\sigma)^2 = 9.5481$ is the power signal to noise ratio. If we expand the bandwidth by a factor of 7/4 the power signal to noise ratio becomes 9.5481 × 4/7 = 5.4561. The voltage signal to noise ratio is therefore 2.336 which gives the new normalised value of $z = 2.336/\sqrt{2}$. The new P_e is therefore 0.0097 from erf tables. A (7,4) block code can correct a maximum of 1 error from the Hamming bound, therefore the probability of a block error is

$$1 - 0.9903^7 - (0.0097)(0.9903)^6 7 = 0.0019$$

Which is lower than the uncoded case of 0.004.

10.3

(a)

error free codewords	0000	0101	1011	1110
correctable errors	1000	1101	0011	0110
	0010	0111	1001	1100
Detectable errors	0100	0001	1111	1010
(appear in more than 1 column)	0001	0100	1010	1111

(b) There are 16 possible single error patterns, four from each codeword. Eight of these error patterns are correctable and the other eight are detectable only.

(c) Probability of a detectable error sequence:

$$= P(1\ error) = 4 \times (0.01)(0.99)^3 = 0.0388119$$

Probability of an undetectable error sequence:

$$= 1 - (0.99)^4 - 4 \times (0.01)(0.99)^3 = 0.00059203$$

10.4

(a)

$$\mathbf{H} = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \Rightarrow \mathbf{G} = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

(b)

codewords						weight
I_1	I_2	I_3	P_1	P_2	P_3	
0	0	0	0	0	0	0
0	0	1	1	0	1	3
0	1	0	1	1	1	4
0	1	1	0	1	0	3
1	0	0	1	1	0	3
1	0	1	0	1	1	4
1	1	0	0	0	1	3
1	1	1	1	0	0	4
						$D_{\min}=3$

(c) This code can correct:

$$\frac{D_{\min} - 1}{2} = 1 \text{ error}$$

(d)

Error pattern	syndrome
000000	000
100000	110
010000	111
001000	101
000100	100
000010	010
000001	001

(e)

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = [111]$$

$$\Rightarrow e = 010000 \Rightarrow c = 111100$$

similarly:

$$r = 000110,\ s = 110,\ c = 100110$$

$$r = 101010,\ s = 001,\ c = 101011$$

10.5

(a)

$$\mathbf{H} = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \Rightarrow \mathbf{G} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

(b)

codewords							weight
I_1	I_2	I_3	I_4	P_1	P_2	P_3	
0	0	0	0	0	0	0	0
0	0	0	1	0	1	1	3
0	0	1	0	1	0	1	3
0	0	1	1	1	1	0	4
0	1	0	0	1	1	0	3
0	1	0	1	1	0	1	4
0	1	1	0	0	1	1	4
0	1	1	1	0	0	0	3
1	0	0	0	1	1	1	4
1	0	0	1	1	0	0	3
1	0	1	0	0	1	0	3
1	0	1	1	0	0	1	4
1	1	0	0	0	0	1	3
1	1	0	1	0	1	0	4
1	1	1	0	1	0	0	4
1	1	1	1	1	1	1	7
							$D_{\min} = 3$

(c) This code can thus correct

$$\frac{D_{\min} - 1}{2} = 1 \text{ error}$$

This code can detect only

$$D_{\min} - 1 = 2 \text{ errors}$$

It cannot do both simultaneously.

(d)

Error pattern	syndrome
0000000	000
1000000	111
0100000	110
0010000	101
0001000	011
0000100	100
0000010	010
0000001	001

The all zero column in **H** would suggest that their is no parity check on some information bits!

(e)

$$s = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} = [010]$$

$$e = 0000010 \Rightarrow c = 1001100 \Rightarrow d = 1001$$

Other double and triple error patterns give same syndromes but always assume that $P(1\ error) \gg P(2\ errors) \gg P(3\ errors)$ etc.

10.6

(i)

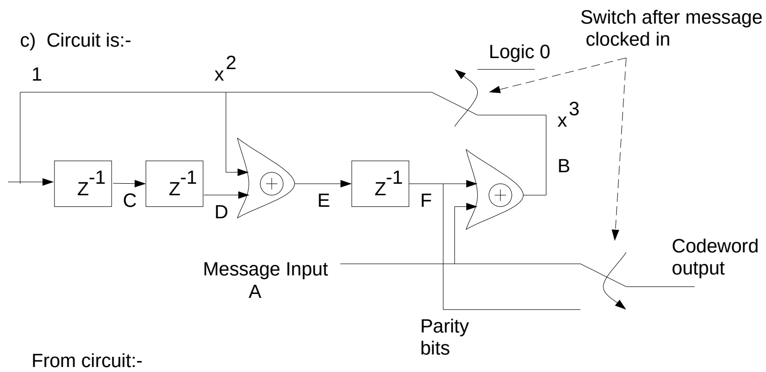
```
a)  1101 | 1000000          1101 | 1010000         Codewords are 1000110
           1101                    1101             and 1010001
           -------                 -------
            101000                  111000
            1101                    1101
            ------                  ------
             11100                    1100
             1101                     1101
             -----                    ----
               110                     001
```

(ii)

```
b)  1101 | 1000110          1101 | 1010001
           1101                    1101
           -------                 -------
            101110                  111001
            1101                    1101
            ------                  ------
             11010                    1101
             1101                     1101
             -----                    ----
                 0                       0
```

(iii) The feedback connections are obtained from the generator polynomial $1 + 0 \times x + x^2 + x^3$. Thus there are no connections $(0 \times x)$ to the output (C) from the first flip-flop (FF) shown below using the MSB first notation:

c) Circuit is:-



From circuit:-

A 1000	A 1010
F⊕A= B 1110	F⊕A= B 1101
previous B C 0111	previous B C 0110
previous C D 0011	previous C D 0011
D⊕ B E 1101	D⊕ B E 1110
previous E F 0110	previous E F 0111

Parity bits are ECB=110 Parity bits are ECB=001

The above tables show the sample by sample results of clocking in the input sequence 0,1,0,1 and evaluation the modulo-two operations.

After opening the feedback connection and clocking the FFs then the first output is E, followed by C and B as these are the corresponding stored digits at the FF inputs and outputs. These digits are then added to the end of the 0101 information digits.

(iv)

```
d)  1101 | 1001110            1101 | 1011001
           1101                       1101
           -------                    -------
            100110                     110001
            1101                       1101
            ------                     ------
             10010                        101
             1101
             -----
              1000
              1101
              ----
               101
```

10.7

This solution uses an **H** matrix with a different first column to that printed in the textbook question!

(a) To calculate the generator matrix we take the first part of the parity check matrix, transpose it and use it as the second part of **G**. The first part is simply the appropriately sized identity matrix.

$$\mathbf{H} = \begin{pmatrix} 1101 & 100 \\ 1110 & 010 \\ 1011 & 001 \end{pmatrix}$$

$$\mathbf{G} = \begin{pmatrix} 1000 & 111 \\ 0100 & 110 \\ 0010 & 011 \\ 0001 & 101 \end{pmatrix}$$

To calculate the codeword for 1110 we multiply the data and the generator matric es

$$\mathbf{d}^T\mathbf{G} = \mathbf{c}^T = (1110)\begin{pmatrix} 1000 & 111 \\ 0100 & 110 \\ 0010 & 011 \\ 0001 & 101 \end{pmatrix} = (1110010)$$

(b) To derive a syndrome decoding table, we simply multiply the transpose of the error pattern **e** with **H**:

Error pattern $\mathbf{e}^T$	Syndrome **s**
0000000	000
1000000	111
0100000	110
0010000	011
0001000	101
0000100	100
0000010	010
0000001	001

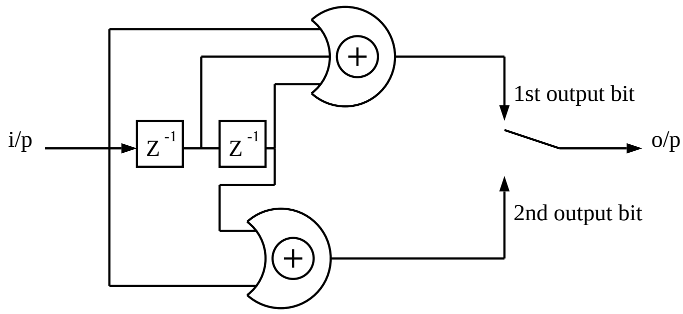
To decode the received codeword 1101110, we simply multiply it by **H** to obt ain the syndrome 010 which means an error in the second last bit. We can simply invert that bit to obtain the error free codeword 1101100 and ignore the last t hree digits to obtain our original data as 1101.

(c) The number of errors a code can correct *t* is given by the Hamming bound, equation (10.4):

$$2^K \le \frac{2^N}{1 + N + {}^NC_2 + {}^NC_3 + \ldots + {}^NC_t}$$

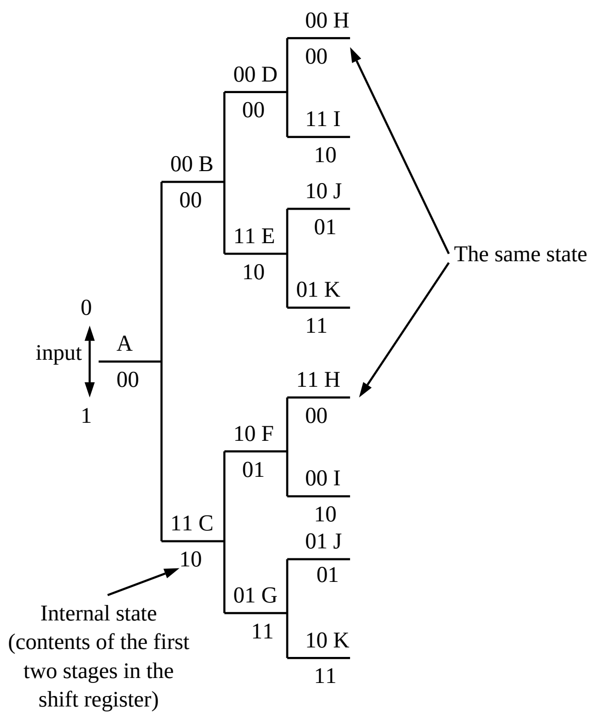
In the (15,11) case with t=1, $1 + N = 16$, where the Hamming bound is an equality, the maximum number of errors a code with these parameters can correct is 1.

10.8

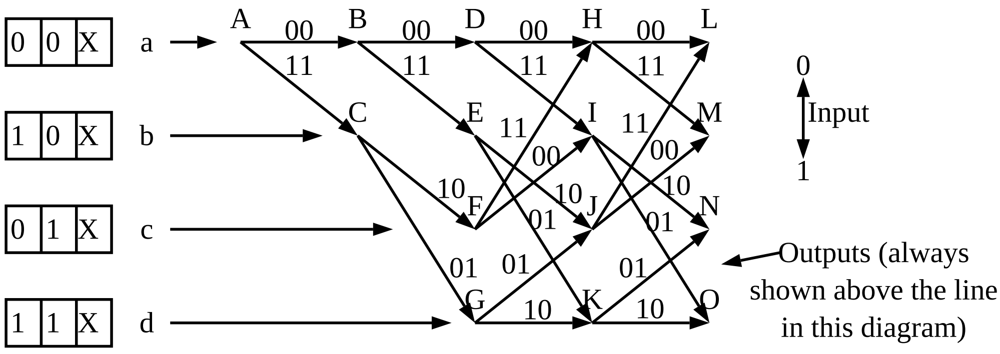


(a)

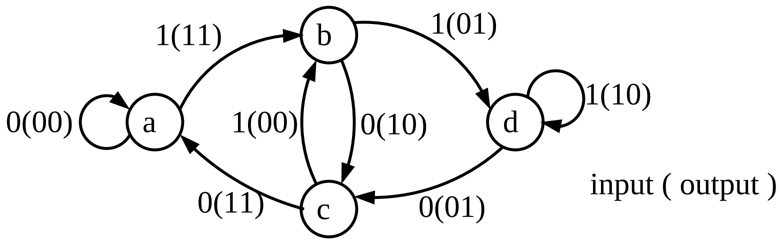
(1)



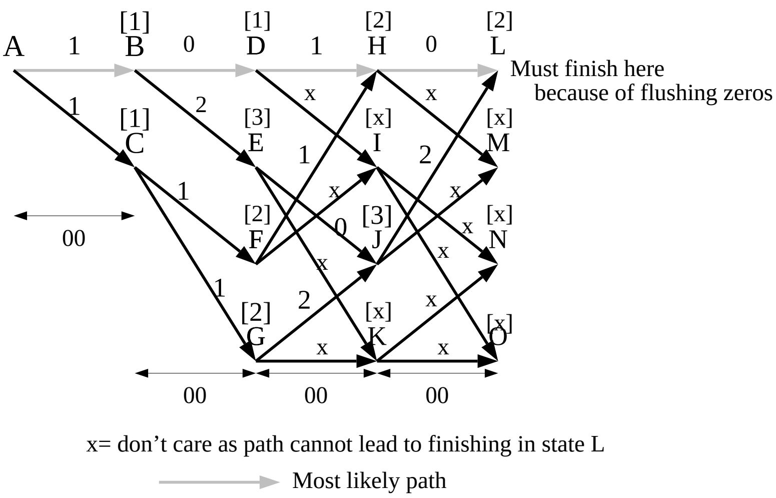
(2)



(3)



(b)

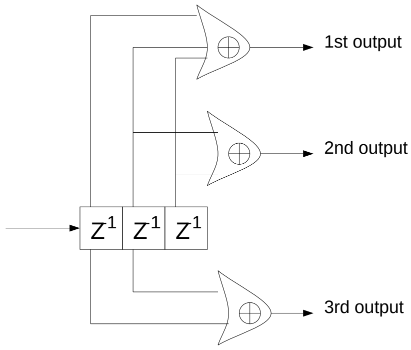


5i

Thus the transmitted symbols were 0, 0, 0, 0 and the data was 0, 0.

10.9

The solution below is for a convolutional encoder defined by: $P_1(x) = 1 + x + x^2$, $P_2(x) = x + x^2$ and, $P_3(x) = 1 + x$:



P1= $x^2 + x + 1$

P2= $x + 1$

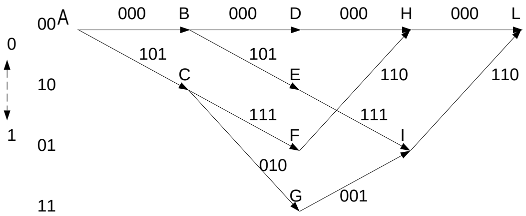
P3= $x^2 + x$

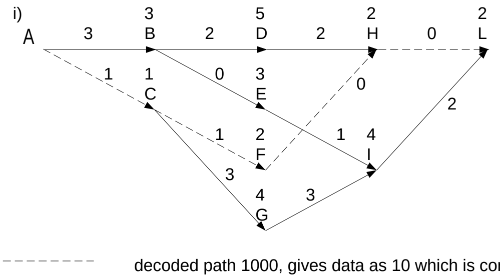
a) The constraint length is 3 and the coding rate 1/3

b)i) Putting a 1 in gives 100 in the encoder which gives 101 at the output.
Then putting a 0 in gives 010 in the encoder which gives 111 at the output.
Then putting a 0 (the first flushing 0) in gives 001 in the encoder which gives 110
Then putting a 0 (the second flushing 0) in gives 000 in the encoder which gives 000
Therefore the encoded sequence is 101 111 110 000

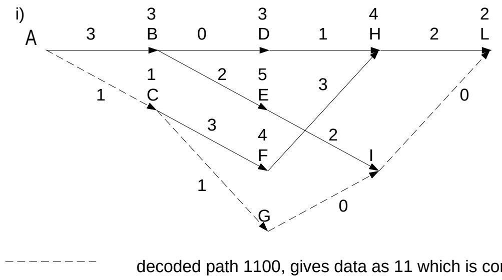
b)ii)
Putting a 1 in gives 100 in the encoder which gives 101 at the output.
Putting a 1 in gives 110 in the encoder which gives 010 at the output.
Putting a 0 in gives 011 in the encoder which gives 001 at the output.
Putting a 0 in gives 001 in the encoder which gives 110 at the output.
Therefore the encoded sequence is 101 010 001 110.

c) The altered sequences are 111 101 110 000 and 111 000 001 110. To decode these sequences, we need the encoding and decoding trellis:





decoded path 1000, gives data as 10 which is correct.



decoded path 1100, gives data as 11 which is correct.

SELECTED PROBLEM SOLUTIONS CHAPTER 11

11.1

$$P_e = \frac{1}{2}\left[1 - \text{erf}\,\frac{1}{\sqrt{2}}\left(\frac{<E>}{N_0}\right)^{1/2}\right]$$

$$\frac{<E>}{N_0} = \left[\sqrt{2}\,\text{erf}^{-1}\,(1 - 2\,P_e\,)\right]^2$$

$$= \left[\sqrt{2}\,\text{erf}^{-1}\,(1 - 2 \times 10^{-6})\right]^2$$

$$= \left[\sqrt{2}\,\text{erf}^{-1}\,(0.999\,998)\right]^2$$

$$= \left[\sqrt{2}\,(3.361)\right]^2 = 22.59$$

$$<E> = 22.59\;N_0$$

$$= 22.59 \times 2 \times 10^{-14}$$

$$= 4.518 \times 10^{-13}\ \text{J}$$

$$T_o = \frac{<E>}{C}$$

$$= \frac{4.518 \times 10^{-13}}{8.0 \times 10^{-9}} = 56.48 \times 10^{-6}\ \text{s}$$

$$R_s = R_b = \frac{1}{T_o} = \frac{1}{56.48 \times 10^{-6}} = 17.7 \times 10^3 \text{ bit/s}$$

11.3

OOK is simplest (non-coherent detector) but has poor SNR performance and detector threshold depends on received signal amplitude. FSK is more complicated, it occupies more bandwidth, but performance is superior as it transmits both 0 and 1 symbols and makes a decision between them. PSK is most sophisticated because it must use coherent detection. However its SNR performance and bandwidth occupancy is superior to other approaches.

For 0 bits centered on 1300 Hz first nulls at ± 600 bit/s rate occur at 700 Hz and 1900 Hz. As 1 symbol carrier is at 1700 Hz then the spacing $2\Delta f$ = 400 Hz and bit rate is 600 Hz. There will thus be some crosstalk between the bits but this is acceptable as the theoretical minimum separation is $2\Delta f = f_b/2$ = 300 Hz.

For 900 bit/s transmission rate with the same system design the upper tone would have to move to $1300 + 900 \times 2\Delta f/600 = 1900$ Hz. The theoretical minimum value for the upper tone is 1750 Hz.

11.8

For FSK $P_e = ½(1 - \text{erf}(E/2N_0)^{½}) = 2 \times 10^{-4}$ or, $\text{erf}\,(E/2N_0)^{½} = 0.9996$.

Using the erf tables then $\left(\frac{E}{2N_0}\right)^{½} = 2.505$.

Received energy $E = 25 \times 10^{-12} \times T_o$

But $N = N_0 B$ where the noise spectral density, $N_0, = 2 \times 10^{-16}$.

$$\left(\frac{E}{2N_0}\right)^{½} = \left(\frac{25 \times 10^{-12} \times T_o}{2 \times 2 \times 10^{-16}}\right)^{½} = 2.505$$

$$6.25 \times 10^4 \times T_o = 6.275$$

$$T_o = 1.0 \times 10^{-4}$$

This corresponds to a 10 kbit/s.

Thus the signal bandwidth is controlled by the data rate or this plus the separation $(f_2 - f_1)$ Hz. For noise purposes it is only the noise associated with the signal bandwidth which enters the receiver and 2 double sided spectra are present.

∴ data rate = 10 kbit/s.

This is broadly commensurate with the 15 kHz spacing between f_2 and f_1 !

For PSK $P_e = \frac{1}{2}\left[1 - \text{erf}\left(\frac{E}{N_0}\right)^{½}\right] = 2 \times 10^{-4}$ or $\text{erf}\left(\frac{E}{N_0}\right)^{½} = 0.9996$.

Using the erf tables then $\left(\frac{E}{N_0}\right)^{½} = 2.505$.

i.e.:

$$\left(\frac{25 \times 10^{-12} \times T_o}{2 \times 10^{-16}}\right)^{½} = 2.505$$

$$12.5 \times 10^4 \times T_o = 6.275$$

$$T_o = 5 \times 10^{-5}$$

This corresponds to the faster 20 kbits/s bit rate, which is twice the value for FSK, as would be expected.

For same bit rate ($\sigma_{PSK}/\sigma_{FSK}$) = ½ as there is only 1 double sided spectra and ∴ ½ the noise present in the PSK receiver!

11.11

$$\eta_s = \frac{\log_2 M}{BT_o}$$

Minimum bandpass BT_o product for ISI reception is given by $BT_o = 1$ i.e.:

$$\eta_s = \frac{\log_2 16}{1} = 4 \text{ bit/s/Hz}$$

$$P_e = 1 - \text{erf}\left[(T_o\ B)^{1/2} \sin\left(\frac{\pi}{M}\right)\left(\frac{C}{N}\right)^{1/2} \right]$$

$$= 1 - \text{erf}\left[\sin\left(\frac{\pi}{16}\right)\left(10^{\frac{24}{10}}\right)^{1/2} \right]$$

$$= 1 - \text{erf}\,[\,(0.1951)\,(15.85)\,]$$

$$= 1 - \text{erf}\,[3.092]$$

$$= 1 - 0.99998773$$

$$= 1.227 \times 10^{-5}$$

$$P_b = \frac{P_e}{\log_2 M} = \frac{1.227 \times 10^{-5}}{\log_2 16} = 3.067 \times 10^{-6}$$